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The Problem of Causality in an Indeterministic Science

HENRY MEHLBERG

The University of Chicago

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Abstract

A differentiation between the principles of causality and strict determinism is suggested, the principle of strict determinism being considered to be incompatible with current scientific theory, while the principle of causality is supported by all contemporary scientific knowledge. In this paper the importance of the principle of causality in indeterministic causal theories is discussed, non-relativistic quantum mechanics being considered in detail. Indeterministic causality and its relation to relativistic quantum theory is also discussed.

1. *The Meaning and Limitations of Deterministic Causality*

In the area of the physical sciences, the claim of strict determinism requires that the value taken on by any measurable, physical quantity for any observable, physical object or system at any time-instant should be accurately predictable on the basis of an appropriate set of quantities whose values at any preassigned, prior instant are accurately known. Loosely speaking, the present determines the future completely. Thus, the observable physical object may consist of two billiard balls on a collision course. The quantity whose value has to be predicted may be the time needed for the collision. The relevant quantities whose present values would suffice to predict the instant of collision may consist of the six spatial coordinates and the six components of the linear momentum of the balls. In conjunction with the laws of Newtonian mechanics, the knowledge of the present positions and momenta will perfectly suffice to predict the exact time of the collision.

Let us imagine, instead of the two billiard balls, a billiard-table where two electrons are used as balls. Some ingenuity may be needed to set up such an electronic billiard game. For instance, the sticks would have to be replaced with electric fields which can be instantaneously turned on to set the electrons in motion. Such technical difficulties would be accidental rather than essential, since their removal would not violate any reasonably established law of nature. One difficulty, however, could not be removed, in view of the fact that, in contrast to standard billiard balls, the electrons are governed by indeterministic laws which do not provide for an accurate prediction of the time of their collision, even if our present knowledge of both electrons should be complete humanly speaking.

To encourage fans of an electronic billiard game, we may point out that those unpredictable electrons are nevertheless governed by causal laws of a special type. The physical laws governing pre-quantal magnitudes dealt with the interconnection of the actual, individual values of these quantities. The causal laws obeyed by electrons deal also with measurable, electronic quantities. But instead of determining the interconnections among the actual values of the relevant quantities, the causal laws of the quantal variety state the interconnections among the probable values of the relevant quantities, or among the 'expectation-values' of these quantities. We could also call the player's attention to other advantages of an electronic billiard game. Thus, hitting the rim of the billiard table, an electronic ball may be scattered powerfully enough to fly back into its own past, as noticed by Feynman (1949a) and Stückelberg (1942). Such an effect is out of the question in billiard games with standard balls.

The concept of causality has survived the collapse of strict determinism in many other ways, more relevant to man's theoretical and practical thinking. The elementary particles in the Brookhaven accelerator change their speed *because* of the electromagnetic field which permeates this research facility. Without this causal relationship, the Brookhaven accelerator would never have been built. The causal relationships involved in this case and inherent in the laws of nature which govern the behavior of elementary particles in electromagnetic fields, obviously serve the same purposes which the deterministic laws of pre-quantal science have enabled man to pursue: to control, to predict with a reasonable accuracy, and to explain the relevant phenomena (Mehlberg, 1958).

Other, significant examples of indeterministic, causal relationships may be recalled. The scintillation-screen glows when hit by impinging electrons *because* of their impact. As discovered by Bethe, the sun radiates *because* energy is released when fusions of hydrogen atoms into helium atoms occur in the presence of carbon atoms inside the sun. The Geiger counter clicks *because* of the passage of ionized particles. A visual sensation arises in a human being *because* his retina was hit by two quanta of light (in the yellow region of the spectrum). This list of significant, indeterministic causal relations can easily be extended.

A differentiation between the principles of causality and of strict determinism is suggested here in view of the present situation in the philosophy of physical science. In the past, the two pricniples were often construed as synonymous. Thus, Kant's formulation of the principle of causality (Kant, 1781) is actually a claim of universal, strict determinism. A differentiation between the two principles is warranted at present, since one of them, the principle of strict determinism, is incompatible with current, scientific theory, while the other one, the principle of indeterministic causality, is strongly supported by the aggregate of all basic, physical theories which constitute the core of contemporary, scientific knowledge. As a matter of fact, this differentiation is compatible with the ordinary usage of the terms 'causality' and 'determinism'. The distinction is not devoid of value even in the area of strictly deterministic, pre-quantal theories. A simple example may illustrate the situation.

If the temperature of a glass of water drops appreciably below 32°F and the water is not completely free of dust-particles and similar centers of crystallization, the water will freeze almost certainly. Under these circumstances, the scientist and the 'man on the street' will not hesitate to consider the drop in temperature as the cause of freezing. However, if the water in the glass happens to be almost completely dust-free, it may occur that it will become 'undercooled' and remain liquid for an appreciable time-interval, in spite of the drop in the temperature. This need not prevent us from considering the drop in the temperature as the cause of freezing, whenever freezing does occur. The event A is then considered to be the cause of the event B if the occurrence of A renders the subsequent occurrence of B considerably more probable. I have used the vague expression 'considerably more probable' on purpose, in order to suggest that indeterministic extensions of the concept of deterministic causality can be effected in several ways. We may feel, for example, that A is causally related to B if the previous occurrence of A modifies in any way the probability of the subsequent occurrence of B, without necessarily bringing the probability of B closer to 1.

The purpose of this investigation is precisely a survey of the main interpretations which can be put on the idea of indeterministic causality, in conjunction with a concise analysis of the status of indeterministic causality in non-relativistic and relativistic quantum theories. Prior to a discussion of these topics, some explanatory remarks concerning the meaning of deterministic causality are called for in this opening section.

The causal relation is usually construed as obtaining between two entities which thereby acquire the title of cause and effect, respectively. Causally related entities were designated as phenomena by classical authors, whereas contemporary thinkers prefer to talk about the causal connection of events or of processes (Wigner, 1964). To avoid estrangement from my contemporaries, I shall construe in the sequel causality as a dyadic relation whose terms are events or processes. What is a physical event? Since our main concern is the status of indeterministic causality in physical sciences, I have to be in keeping with linguistic usage in the group of physical sciences, and, in particular, with the subgroup of quantal sciences which are mainly responsible for the new attitude towards causality. Both events and processes are occasionally referred to in quantal sciences. But the ideas which specifically characterize the physicist's new interpretation of causality are usually expressed in terms of measurable, physical quantities, called 'observables' by those who speak Dirac's language (Dirac, 1935). That an event E occurs, means that a measurable quantity Q associated with E takes on a particular value q for an observable physical object or system s at a particular time t. What do we call a measurable physical quantity in this context?

This happens to be an embarrassing question. To realize this, it suffices to notice that the highly problematic concept of a physical property P, attributable to the physical object or system s, can be equivalently replaced with a characteristic, associated quantity Q_P. Instead of asserting that the system s has the physical property P, we may state equivalently that the quantity Q_P takes on the value 1 for the system s. Similarly, the absence of the property P from the system s corresponds to the value 0 of Q_P. Thus, every physical property P, whether or not literally quantitative, can be replaced with the corresponding quantity Q_P. Conversely, all measurable physical quantities can be construed as aggregates of appropriately related physical properties. The embarrassing nature of the question concerning the meaning of a measurable, physical quantity, and its relevance to the clarification of deterministic causality, can be more easily explained if the concept of physical property is used instead.

We may say, tentatively, that the event E, constituted by the fact that the physical system s had the property P at the time t, was the cause of a subsequent event E' consisting in another triplet (s',P',t') if every reproduction of E is always followed immediately by a reproduction of E'. This is a somewhat more precise formulation of Hume's memorable analysis of the idea of deterministic causality in terms of regular, temporal succession (Hume, 1748). However, Hume and all his successors failed to realize the embarrassing nature of the concept of property involved in Hume's clarification of deterministic causality.

'Property' is a purely logical concept, associated in the familiar way with the concept of class or set. The Humean definition of a deterministic, causal relation, reworded in terms of classes, would read as follows:

> The fact that the system s belonged to the class C at the time t was the cause of the fact that the system s' belonged to the class C' at a later time t' means that the membership of any system s'' in the class C at any time t'' is always immediately followed by the membership of a system s''' in the class C' at an instant following t''.

This definition, however, although somewhat more precise than Hume's initial formulation, is tautologically satisfied by any two unit-classes U, U' provided that the membership of the only element of U be immediately followed by the membership of the sole element of U'. Hence, Hume's interpretation of deterministic causality in terms of regular succession is literally untenable since it implies that every single event A is the cause of every subsequent, single event E'.

The above difficulty inherent in the Humean definition of deterministic causality cannot be adequately analyzed in the context of the present investigation. For our purpose, it may suffice to indicate a procedure which can be used to overcome this difficulty. The procedure consists in duplicating another procedure which Bertrand Russell has once used in his philosophy of logic He suggested to define a law of logic as any true statement whose formulation requires, apart from the use of some variables, only 'logical constants'. In turn, to explain the meaning of a 'logical constant', he made use of the remarkable achievement of symbolic logic which consists in the effective definability of the class of all logical constants by means of an extremely small number of such constants, namely, the number 2. Accordingly, the class of all logical constants can be identified with the class of all terms that are effectively definable by means of two appropriate, logical terms, e.g., Sheffer's stroke and the existential quantifier. Finally, these two undefined constants of logical theory are, obviously, definable in logical meta-theory (Kleene, 1962), namely, by explicit enumeration.

A similar, most remarkable result, has also been obtained in physical science. All physical concepts required for the formulation of all major physical theories are effectively definable in terms of a short list of physical quantities $Q_1, Q_2, \ldots, Q_N$, where each quantity Q_i is either a scalar-valued, or a vector-valued or a tensor-valued, or an operator-valued function definable over any set of spatiotemporal coordinates which can be bi-uniquely mapped into all the point-like elements of the space–time continuum. The N undefined concepts of physical theory are again definable by explicit enumeration in the physical meta-theory. By the same token, it is possible effectively to define in physical meta-theory the concepts of any physical quantity, or property, or system, or event, or process. The relation of deterministic causality among physical events can therefore be effectively defined in physical meta-theory, by resorting to the same procedure. Obviously, the N undefined concepts of physical theory, definable by enumeration in the meta-theory, do not correspond to properties associated with unit-classes.

The above remarks about the reducibility of all significant, physical quantities to a short list of basic quantities susceptible to an effective definition by enumeration in physical meta-theory, explain why I prefer to limit my analysis of indeterministic causality to the area of physical sciences. Otherwise, the vital concepts of event, property, object or system, etc. would be devoid of any precise meaning. This would also make it impossible soundly to define the concept of causal relation, either deterministic or indeterministic. The restriction to the physical area is probably less crippling than it might seem. For there is good evidence, at present, that neither chemistry nor biology transcend the conceptual scope of physics. Only the area of psychological sciences might be affected to the extent to which causal relations have already been firmly established in this area.

Let me finally mention that, according to the aforementioned British thinker, Bertrand Russell, only the professional philosophers and the uncivilized inhabitants of the Fiji Archipelago firmly believe that the occurrence of an event E could make the occurrence of a subsequent event E' necessary (Russell, 1927). According to Russell, these two human groups entertain their common view of the nature of the causal relation among consecutive events because they both overlook the possibility that, after the occurrence of the earlier event E, some third event E'' may still prevent E' from happening. It is also obvious, however, that Russell has simply overlooked the fact that, according to Einstein's Special Theory of Relativity, espoused by Russell, nothing could possibly prevent E' from happening after the occurrence of E in those numerous, factual situations where, to keep E' from happening, something would have to travel with a speed exceeding that of light.

The same point can be made somewhat differently. The causal laws governing the electromagnetic field have been formulated by Maxwell in a system of partial, differential equations. In mathematical analysis, Maxwell's equations are classified under the heading of 'hyperbolic' differential equations (Courant & Hilbert, 1965). It is a mathematical fact that the existence and uniqueness of a solution to a system of partial, differential equations of the hyperbolic variety is guaranteed by an appropriate set of initial and boundary-conditions. In the case of the electromagnetic field, the initial conditions specify the state of this field at some instant t_0 and the boundary-conditions become redundant, since the instantaneous state of the entire space is involved in the initial condition. Accordingly, on the assumption that the initial condition includes the event E, nothing could possibly prevent the event E' from happening at a later instant.

2. *Indeterministic Extensions of Deterministic Causality*

To apply the principle of causality to indeterministic, causal theories, we must first redefine the concept of deterministic causality discussed in the opening section in order to make the redefined causal relation meaningful both within and without the aggregate of all quantal theories. One way of redefining this concept is to construe the extended causal relation as obtaining between two consecutive events E, E', whenever the conditional probability that E' will occur on the assumption of a previous occurrence of E is larger than the conditional probability of E' on the assumption that E failed to occur prior to E'. The meaning of the term 'event' was clarified in the opening section. A relation of deterministic causality would then obtain between E and E' if the conditional probability of E' given the previous occurrence of E has the special value 1, corresponding to certainty.

In most cases of interest in the quantal area, the physical event E considered as a cause of E', includes, in addition to the particular value q which the quantity Q associated with E takes on, at the time t, for the particular system s in which E happens, the assumption that this system s was at this time t in a particular, specifiable quantum state R (Wigner, 1967). The idea of quantum state was considered, for important reasons, by Dirac as the principal, conceptual innovation of the whole of quantum theory. Nevertheless, this idea can be rigorously derived from the notion of a measurable quantity Q whose values are ascribable to a physical system s, in the following way.

(1) We first stipulate, by definition, that the quantities Q_1, Q_2, ..., Q_n are *compatible* if all of them can take on particular values for the same system s at the same time t.

(2) A compatible set of quantities Q_1, Q_2, ..., Q_n is called *maximal* if the value of every quantity Q taken on by the system s at the same time t is a single-valued function of the particular values of this compatible set of quantities for the same system at the same time.

(3) The quantum state R of the system s at the time t can be identified, by definition, with a set of particular values of a maximal and compatible set of quantities, provided that all these values be attributable to the same system at the same time.

There is a second, possibly most important, indeterministic extension of causality which is associated with the notion of 'interacting physical systems'. This phrase appears frequently in almost every major presentation of every quantum theory. The idea of causal interaction, considered by Kant (1781) as a fundamental category of the human mind, was then rejected by Schopenhauer (1814), who felt that causal interaction is incompatible with the temporal succession of cause and effect, hardly ever denied in philosophical literature, both before and after Hume. Yet, causal interaction, often illustrated by the scattering of colliding particles, is apparently inherent in every contemporary, quantal theory. To avoid Schopenhauer's objections, we may assume, by definition, that two particles or two systems of particles interact with each other during the time-interval (t_1, t_2) if the effect of this interaction is different from the superposition of the two effects which each system would have brought about after the interaction-interval should the other system not have been present.

A somewhat more technical definition of causal interaction may come closer to what is on the physicist's mind. Suppose that the energy of the system s_1 is represented by the Hamiltonian operator H_1 and that this holds also of the system s_2 and the Hamiltonian H_2. The energy of the compound system (s_1, s_2) made up of the two aforementioned systems will be represented by a third Hamiltonian H_3. Under these circumstances, three cases are possible: (i) H_3 is equal to the sum of H_1 and H_2. (ii) H_3 is the sum of three terms, namely, H_1, H_2, and the so-called interaction-Hamiltonian of the compound system. (iii) During the interaction-interval, neither the system s_1 nor the system s_2 have Hamiltonian operators of their own although the compound system has a Hamiltonian operator which represents its energy. In the first case, we shall stipulate, by definition, that the two component systems do not interact. If either the second, or the third case materialize, the two component systems will be said to interact with each other, by definition.

It can easily be verified that the above definition of causal interaction is not open to Schopenhauer's criticism. A more important consequence of the proposed definition is that the replacement of deterministic causality with a precisely definable type of indeterministic causality becomes inevitable for interacting systems. The point is, that the temporal evolution of a quantum mechanical system is determined by the so-called Schrödinger's time-dependent, partial differential equation which interrelates any two consecutive quantum states of this system. Since Schrödinger's equation is of first-order with regard to time, its solution is uniquely determined by the initial condition, i.e. the quantum state of the system at an instant t_0. Consequently, the successive states of a quantum mechanical system obey a strictly deterministic law.

This conclusion holds, however, only on the assumption that the system under consideration did not causally interact with any other quantum mechanical system during a time-interval, however short. For, if such an interaction did occur, then the quantum mechanical system would have interacted with another system in one of the two ways described in the preceding paragraphs of this section. If the interaction was of the first type, then the system has a Hamiltonian operator of its own but Schrödinger's deterministic law would not be applicable to it, because of the presence of an interaction-Hamiltonian which conflicts with the validity of Schrödinger's equation. On the other hand, should the interaction be of the second type, then the system would have no Hamiltonian of its own during the interaction-interval. Consequently, the system would not have obeyed Schrödinger's law, because the latter holds only in situations when the relevant Hamiltonian does exist.

The collapse of strict determinism for the system during the interaction-interval would imply that indeterministic transitions between the consecutive states of the system would occur during the interaction-interval. This circumstance would also affect any quantum state of the system after the interaction had subsided, because the indeterministic transitions during the interaction would obviously affect every future state.

To sum up. The indeterministic extensions of the concept of deterministic causality can be effected in any quantum theory in several, significantly different ways. Two of these extensions lead to an indeterministic causal relation which obtains between individual quantum-events and does not imply the concept of the expectation-value of a physical quantity in a statistically definable aggregate of physical systems. In the sequel, I shall refer to these two extensions as cases of 'individual, indeterministic causality'. These two cases involve either the replacement of a deterministic connection between the events E_1 and E_2 by the conditional probability of the occurrence of E_1 relative to a previous occurrence of E_2, or the assumption that E_1 and E_2 take place in a system s whose interaction with any other system cannot be accounted for by an appropriate, time-dependent Hamiltonian of the system s.

3. *Causality in Non-relativistic Quantum Mechanics*

In the preceding section, I have discussed various indeterministic extensions of the concept of deterministic causality. We shall now try to find out what kind of causality-principle holds in one crucial, indeterministic theory, namely, non-relativistic quantum mechanics. However, prior to this investigation, we have to comment on one specific aspect of quantal causality. The point is, that some indeterministic extensions of deterministic causality deal with causal relations among those special conditions of systems obeying quantal laws which are called quantum states. We did agree that causal relations will be construed in the sequel of this paper as obtaining among events. In what sense are the quantum states (Wigner, 1967) of physical objects to be considered as events?

In Section 2, the quantum state of a physical object or system s at a time t was identified, by definition, with the set of particular values taken on by a maximal and compatible set of measurable quantities for the system s at time t. On the other hand, an event was defined as the fact that a measurable quantity Q takes on, for the system s at the time t, some particular value q. Consequently, a quantum state ascribable to a system s at a time t can be construed as a compound event involving a single system s at a time t in addition to the circumstance that not one, single quantity, but a finite, maximal and compatible set of quantities taking on particular values for the system s at the time t is implied. There is no objection to classifying such compound events under the general heading of events. The causal relation among quantum states would then be a special case of a causal relation obtaining among events.

On the other hand, I would like to remind the reader that quantum mechanics admits many mathematical models which are usually called representations or pictures of quantum mechanics. More specifically, I have referred a few times to the so-called Schrödinger representation of quantum mechanics. In this representation, which dominates von Neumann's pioneering work, the quantum state of a physical system corresponds to a complex-valued, square-integrable function Ψ of the spatio-temporal coordinates of all the point-like components of the system. The fact that the quantum state of a system s at the time t is represented, in the Schrödinger picture, by a complex function Ψ, has misled several investigators who doubted whether a complex function is a measurable quantity at all. Consequently, a quantum state described by such a function could not be a constituent of a physical event, since each event consists in the attributability, to a system s at a time t, of the particular values taken on by a finite set of measurable quantities for this system at this time.

This doubt as to whether quantum states are classifiable under the heading of events, turns out to be unjustified even if all the quantum states are considered in their Schrödinger representation. One finds, on closer analysis, that there is a bi-unique correspondence between the class of all quantum states and the class of pairs of two real quantities, each pair consisting of the square of the absolute value of the function Ψ and of the rate of change of the square of the modulus of this complex function. The point is that, in quantum mechanics, the square of the modulus of the function Ψ is interpreted as the probability that the physical system in the quantum state describable by Ψ can be found at a particular space–time point (or, in a collection of space–time points when compound systems are involved) whose spatio-temporal coordinates are the values of the arguments on which the function Ψ depends. The probability of the presence of a physical system in a specified aggregate of space–time points is a measurable quantity, since its value can be determined by spatio-temporal measurements in conjunction with the observation of the location of all the components of the physical system in a set of space-time points. If the probability of a localizable presence of a physical system can be quantitatively evaluated then this holds also of the rate of change of this probability. However, these two measurable quantities, namely the probability of a specified spatio-temporal location and the rate of change of this probability, determine uniquely the complex function Ψ and are also uniquely determined by Ψ (Feenberg, 1933). Thus, attributing to a system s at a time t a quantum state represented by the complex-valued function Ψ in Schrödinger's picture of quantum mechanics comes actually to stating that this system is involved at the time t in a compound event. Consequently, regardless of whether we define quantum states in terms of maximal and compatible sets of measurable quantities or describe these states by the complex-valued functions of the Schrödinger representation, we shall always be justified in investigating the causal relations among quantum states, since the latter are events of a specifiable type.

I shall now attempt to find out what kind of causality-principle, if any, holds in non-relativistic quantum mechanics. In this context, the causal principle is construed as the claim that every quantum mechanical event has a cause. The 'kind of causality-principle' for which we are looking, depends upon whether a deterministic causal relation is asserted in the principle, or any of the indeterministic extensions of the concept of deterministic causality is involved in the causal principle. Obviously, the fact that the principle of deterministic causality applies to the quantum states of those quantum mechanical systems which never interacted with any other system, does not substantiate the validity of any principle of causality in quantum mechanics. For a principle of causality in this area requires that every quantum mechanical event should have a cause. All which can be shown is that the principle of deterministic causality applies to the infinitesimal area of quantum mechanical events which consists of quantum states of interaction-free systems.

In the next section, we shall see that in a specifiable class of quantal situations, the conditional probability of any event to occur at the time t on the assumption that another event happened at an earlier instant is never equal to 1, although it exceeds the conditional probability of the later event on the assumption that the earlier one failed to materialize. This is always the case when an excited atom surrounded by a field of electromagnetic radiation both emits a quantum of radiation and becomes less excited. The causal relation between the events of excitation and emission is obviously indeterministic since the excited state of a radiating atom could also ensue in other emissions (unless excitation of the the atom happened to be minimal). Moreover, the atom might also have persevered in its excited state or become more excited by absorbing a quantum of radiation from the surrounding field.

It must be granted that in such cases of emission, absorption and persisting excitation of an atom located within a field of electromagnetic radiation, the atom is obviously interacting with the radiation-field. We have seen that whenever transitions between consecutive quantum states of a system which interacts with other systems occur, the principle of deterministic causality no longer applies. However, in the aforementioned cases of exchange between an atom and a radiation-field, the causal relation obtained between two events none of which was a quantum state of a system. Moreover, one of these events, namely the excitation of a compound or point-like physical system, may also go on for an indefinite period of time, in contrast to the indeterministic causal transitions among quantum states where a continuous change of the quantum state under consideration follows from the relevant causal laws, regardless of whether the causality involved is deterministic or indeterministic.

We thus realize that the indeterministic extensions of deterministic causality apply both to quantum states of physical systems and to events which are not comprehensive enough to constitute a quantum state of physical systems. The significant difference between the indeterministic causality of events involving an exchange of energy between a localizable, physical system and the surrounding radiation-field and the relation of indeterministic causality which obtains between two events occurring in the same particle-system, will be described in the sequel. Before discussing this difference, we must realize that a full treatment of the indeterministic causality-principle which governs the energy-exchanges between excited systems and the surrounding radiation-field belongs actually in quantum electrodynamics (Akhiezer & Beresteckii, 1965), which is a relativistic quantum theory to be dealt with in the next section. In this section, I wanted mainly to emphasize the specific breakdown of deterministic causality which takes place when one of the causally related events occurs in a particle-system while the other event happens in the surrounding radiation-field. This gave us also the opportunity of establishing relations of indeterministic causality among events which are neither quantum states nor are consequences of indeterministic transitions among quantum states.

The latter case can be illustrated by indeterministic transitions between the quantum states of a system which interacts with another particle-system. Suppose that, in view of the interaction of the system s_1 with another system, the transitions between the consecutive states of s_1 are governed by an indeterministic causality-law. According to this law, the conditional probability of a specific state R_1 of s_1 at the time t_1 has a positive conditional probability less than 1, on the assumption that at the earlier time t_2 the state of s_1 was R_2. To be different and not to be connected by a deterministic law, these two states must be uniquely defined by sets of particular values of two maximal and compatible aggregates of quantities such that at least one quantity Q occurs only on one of these aggregates, e.g. in state R_1.

Under these assumptions, the quantity Q has a definite value at the time t_1 for the system s_1 while, at the time t_2, the value of Q for s_1 is non-existent. There are therefore two consecutive events occurring in the system s_1, namely the particular value of Q at the time t_1 and the non-existence of any value of Q at the time t_2. Obviously, the conditional probability of the first event on the assumption that the second occurred at the time t_2 is exactly equal to the conditional probability of the quantum state of s_1 at the time t_1 on the assumption that, at t_2, the other of the aforementioned quantum states materialized. We thus realize that the indeterministic extension of the causality relation, based on the idea of conditional probability and applied to events which are not quantum states, falls under two categories. The first can be illustrated by the causal laws which govern the exchanges of energy between excited atoms and the surrounding radiation-field. The second is exemplified by the indeterministic causal laws obeyed by events occurring in the same quantum mechanical system. The quantum electrodynamic laws, to be discussed in more detail in the following section, establish relations of indeterministic causality among events classifiable under the first category. The most distinctive feature of these causally related events is that they influence each other independently of the causal laws which govern the quantum states of the compound, quantum electrodynamic system consisting of the entire electromagnetic field and all the charged matter in the field. The indeterministic causal relations among quantum mechanical events exemplify the second category.

I shall now discuss briefly the relation between the *indeterminism* of quantum mechanical processes and the *indeterminacy* of certain quantum mechanical quantities, the latter being usually referred to as Heisenberg's Uncertainty Principle. Von Neumann, whose contribution to the clarification of quantum mechanics was second to none (to say the least), has nevertheless twisted the issue of indeterminism in quantum mechanics in a most peculiar way. His treatment of this issue obviously remains extremely important, because it still dominates the scientific outlook and, also, because it has originated with von Neumann. Von Neumann's approach to non-relativistic quantum mechanics was based almost exclusively on the Schrödinger representation of this theory. He discussed quantum mechanical indeterminism in terms of what he called 'pure ensembles of quantum mechanical systems', i.e. ensembles of non-interacting systems all characterized by the same quantum state, or, equivalently, by the same complex-valued and square integrable function Ψ whose arguments are the spatio-temporal coordinates of all the components of the quantum system under consideration.

In identifying quantum mechanical indeterminism and indeterminacy, it is possible that he had in mind the following reasoning. Since the complex, square integrable function Ψ describing completely the instantaneous quantum state of the pure ensemble, contains the maximum of physical information which could possibly be obtained about this ensemble, any question concerning the particular value of a quantity with a positive standard-deviation in the ensemble could not be answered. Thus, the maximal information about the ensemble would not suffice to determine the correct answer to a question about the particular value of a physical quantity of the aforementioned type. If this was the reason for von Neumann's identification of quantum mechanical indeterminism with the non-existence of the type of pure ensemble which I have just mentioned, then the simplest refutation of his view would refer to the fact that all questions about the particular values of quantities whose standard-deviation is positive are actually answerable: a physical quantity with a positive standard-deviation in a given pure ensemble has no particular value under these circumstances. The above question can therefore be answered by pointing out the non-existence of any particular value of a quantity with a positive standard-deviation.

It is possible, however, that, in identifying indeterminism with the non-existence of pure ensembles with vanishing standard-deviations of all measurable quantities, von Neumann had another aspect of the situation in his mind: he may have felt that quantum mechanical indeterminism is synonymous with the non-existence of pure ensembles with vanishing standard-deviations of all physical quantities, because the question concerning the outcome of a successful measurement of a quantity with positive standard-deviation in the present quantum-state of the ensemble cannot be answered on the basis of an accurate knowledge of this quantum state, despite the fact that such knowledge contains the maximum information about the present state which could possibly be obtained, or even exist. This alternative, hypothetical reason which may have induced von Neumann to identify indeterminism with the non-existence of a specifiable type of pure ensemble would make his identification more plausible. On closer analysis, this hypothetical, second argument is as untenable as the first.

4. *Indeterministic Causality in Relativistic Quantum Theories*

Several relativistic quantum theories (Wigner, 1957; Mehlberg, 1966; Greenbaum, 1963) are now reasonably supported by available, observational evidence, for instance quantum electrodynamics and the quantum theory of dispersion (Goldberger, 1960). The imperative need for constructing relativistically covariant quantal theories has become evident ever since elementary particles whose characteristics make a relativistic treatment inevitable came into the picture. Thus, a non-Einsteinian, e.g., Newtonian, approximation in the laws of nature which govern particles moving with the speed of light or any comparable velocity, is out of the question. Hence, a non-relativistic quantum mechanics of light-quanta or photons, was never considered. Similarly, a quantal treatment of elementary particles susceptible to annihilation or to creation could not be successful, unless it were relativistic. As a matter of fact, most of the elementary particles discovered so far can be created or destroyed under specifiable circumstances and are, consequently, describable only by relativistically covariant laws of nature. I shall not list other properties of elementary particles which could be dealt with only in relativistic quantum theories. They are numerous and most significant, and there is little doubt that the most important experimental discoveries of our century are related to elementary particles. These discoveries explain why relativistic quantum theories have become indispensable.

I propose to study in this section a specific version of the indeterministic principle of causality which applies to one representative, relativistic quantum theory, namely quantum electrodynamics.

Let us start with an outline of quantum electrodynamics, sufficiently accurate for a discussion of the status of causality in this theory. The conventional formulation of quantum electrodynamics (Akhiezer & Beresteckii, 1965), in most major treatises on this subject, begins with a list of Maxwell's equations of the electromagnetic field, supplemented by Dirac's equations of the relativistic quantum mechanics of electrons and positrons. Maxwell's equations are subsequently reformulated, in a familiar way, in terms of the vector and scalar potentials of the electromagnetic field. A decisive, conceptual transformation is then performed upon both the reformulated equations of Maxwell and Dirac's relativistic, quantum mechanical equations. Namely, the two electromagnetic potentials, initially construed as real-valued functions over the spatio-temporal continuum, are now reinterpreted as operators which still depend upon spatio-temporal coordinates but act on the quantum state of the compound quantal system made up of the electromagnetic radiation-field and all the electrons and positrons interacting with this radiation. Similarly, Dirac's equations, initially interpreted as descriptions of the behavior of spinors (i.e., of four-component functions of the spatio-temporal coordinates) associated with each electrically charged particle in Dirac's relativistic quantum mechanics, are also reinterpreted as operators acting on the state of the aforementioned, compound quantum electrodynamic system. This reformulation and reinterpretation of both Maxwell's and Dirac's equations constitutes the conventional quantum electrodynamics.

However, in order to discuss the problems of causality in this relativistic, quantal theory, it seems preferable to replace the aforementioned, standard equations of quantum electrodynamics with an alternative, elegant formulation developed by Feynman (Feynman & Hibbs, 1965). He followed the example set by major classical theories of physics, each of which was derivable from a 'Least Action Principle'. This holds, for example, in Newtonian mechanics and Maxwellian electrodynamics. We shall see that the Least Action Principle in Feynman's formulation of quantum electrodynamics is used by him in a significantly different way which sheds a new light on indeterministic causality in quantum electrodynamics and, possibly, in other quantum theories, including the non-relativistic theory of quantum mechanics.

Let me first recall that the Least Action Principle, if used in a classical, pre-quantal theory, comes to the requirement that any real process described by the theory and providing a specific transition from a preassigned initial stage to a preassigned final stage, differs in one, crucial respect from other, conceivable processes having the same initial and final stages. The point is, that the physical action involved in the real process is smaller than the action required by all other processes.

In this context, the 'physical action involved in a transitional process' is defined, in the simple case when the same amount of energy is emitted or absorbed at each phase of the process, as the product of the duration of the process and the Lagrangian operator changing possibly throughout the entire process. If the energy characterizing each phase of the process is variable and both emission and absorption of energy are taken into account, then the physical action involved in a transitional process is defined in a somewhat more technical way, namely as the time-integral of the Lagrangian computed between the initial and final instant of the process. The principle of strict determinism inherent in the classical theory under consideration, for example in Newtonian mechanics, reads: the real process which provides a transition from the initial to the final state of a mechanical system governed by the laws of Newtonian mechanics differs from any other process with the same initial and final stage of the same mechanical system in one, decisive respect: the physical action involved in the real process is smaller than the action that any other transitional process, with the same initial and final stage of the same mechanical system would involve.

The initial and final stage of a process are often referred to as its 'boundary-conditions' (Feynman & Hibbs, 1965). In this terminology, we may say that Newtonian mechanics is deterministic because every transitional, mechanical process is uniquely determined by its boundary-conditions. This is not the usual formulation of the principle of determinism, which is construed as the claim that the initial state of a mechanical system which has not interacted with any other systems, uniquely determines all the future states of this system. It is easy to find out that the concept of initial state of a mechanical system which occurs in the formulation of determinism in terms of the boundary-conditions differs significantly from the concept of initial state involved in the familiar formulation of determinism. Once this ambiguity is taken into account, the two formulations of strict determinism in Newtonian mechanics can be shown to be logically equivalent and inter-deducible. The question whether a similar equivalence obtains between the two possible formulations of the quantum electrodynamic principle of indeterministic causality will be only mentioned in the sequel. A more accurate analysis of quantum electrodynamics would show that, in this theoretical area, the two formulations of the principle of causality are no longer logically equivalent.

I shall now briefly explain the main ideas involved in Feynman's quantum-electrodynamic Least Action Principle. Since this theory explores the interaction between electromagnetic radiation and electrically charged particles, we must anticipate that, in addition to the energy of the radiation and of matter interacting with the radiation, there is a distinct interaction-energy, not identifiable with either the former or the latter energy. Consequently, the total action involved in a transitional process is the sum of three component actions.

(1) The action S_{rad} which would be performed in the field of electromagnetic radiation even if no matter were present.

(2) The action S_{mat} generated by the energy of the electrons and positrons in the field, regardless of S_{rad}.

(3) The action S_{int} due to the interaction-energy of the electromagnetic field with the charged matter in the field. The total action S of the compound system consisting of the interacting electromagnetic field and the electrons and positrons in the field obviously satisfies the equation $S = S_{rad} + S_{mat} + S_{int}$.

The fundamental difference between the classical handling of the Least Action Principle and its use by Feynman can now be explained as follows. In all classical theories of physics, the minimum requirement on the action involved in a transition from a preassigned initial state to a preassigned final state of the physical system under consideration was used to uniquely determine the process which effects this transition. In Feynman's axiomatization of quantum electrodynamics, the minimum requirement on the action S serves completely different purposes, the most important of which is the derivation of Maxwell's equations for the electromagnetic field from the assumption of a minimal action being involved in the transition from the initial to the final state of the field. On the other hand, the transition between the initial and final state of the compound system made up of radiation and charged matter does not depend, at all, on the *extremal value* of the action involved in this transition. Actually, the only axiom of Feynman's quantum electrodynamics (Feynman & Hibbs, 1965) states that the probability-amplitude involved in any conceivable transition from the initial to the final state of the compound system is proportional to the specific action which such a transition would require. In this context, a probability-amplitude for any specific transition is a complex quantity whose squared modulus is equal to the probability of the transition. Consequently, the process effecting a transition from an initial to the final state is not determined uniquely by these two states or boundary conditions. Actually, every process effecting this transition makes an equal contribution to the total probability of getting from the initial to the final state of the compound system.

This novel use of the Least Action Principle in connection with the transition from an initial to a final state of the compound system explains why a theory derivable from a Least Action Principle may be deterministic but need not be so. This clarification of the indeterministic causality principle which governs the transitions between the consecutive states of the compound quantum electromagnetic system, seems to me to be the philosophically most significant achievement of Feynman's approach to quantum electrodynamics. To some extent, the switch from deterministic causality in classical electrodynamics to the indeterministic causality of quantum electrodynamics resembles the switch from classical determinism in Newtonian mechanics to the indeterminism of non-relativistic quantum mechanics. However, with regard to the scope of deterministic causality, the fundamental difference between non-relativistic quantum mechanics and relativistic quantum electrodynamics is conspicuous. In quantum mechanics, the determinism of all systems which never interacted with any other system is a direct consequence of Schrödinger's time-dependent equation. In quantum electrodynamics, the indeterministic causality of the compound system is a direct consequence of the basic axioms, although the compound system could not possibly interact with any other system.

It is natural to expect that the relativistic invariance of quantum electrodynamics will entail the resemblance of the causality-principle of this theory and the causality-principles of other relativistic theories, either classical or quantal. This is actually the case. Thus, one of the most striking consequences of the Special Theory of Relativity was that the speed of light *in vacuo* is also the maximal speed of causal propagation. In other words, no events occurring, respectively, at the points P_1 and P_2 on Minkowski's space–time could be causally related unless these two points can be connected by a light-signal. The relevant passage in Heitler's presentation of quantum electrodynamics (Heitler, 1960) reads: 'The field-strengths [namely electromagnetic] at two points of space–time which cannot be connected by light, commute with each other.' We may add that the electric and magnetic field-strengths being construed as operators working on the state of the compound system are assumed as commuting with each other at two points of relativistic space–time because this commutativity relation amounts to the possibility of making two independent measurements on the values of these operators at the above two spatio-temporal points. The possibility of performing these two independent measurements at the space–time points P_1 and P_2 is equivalent to the impossibility that two events occurring at the points P_1 and P_2, respectively, can be causally related to each other.

It remains to comment briefly on the somewhat modified meanings of the terms quantum state, quantum event, quantum process and the boundary conditions of a quantum process in the context of quantum electrodynamical theory. In the case of non-relativistic quantum mechanics, we have presented two equivalent definitions of a quantum state: it could then be either identified with a set of particular values taken on, for the system under consideration, by a maximal sequence of compatible quantities, or as a condition of the system described in the Schrödinger representation of quantum mechanics by a complex, square-integrable function of the spatio-temporal coordinates of all the components of this system. In the case of quantum electrodynamics, the meaning of a quantum state of the compound system consisting of electromagnetic radiation and the charged matter interacting with this radiation, is more intricate. Perhaps the simplest way of understanding this new meaning consists in realizing that, in quantum electrodynamics, a quantum state is uniquely determined by the particular eigenvalues of all the 'occupation-number operators' defined for this compound system. Thus, there are four types of electromagnetic quanta or photons (Heitler, 1960), each type being susceptible to a further specification by the indication of the energy, linear momentum, polarization and spin of any particular photon. A particular eigenvalue of the occupation-number operator associated with a particular type of photon, including its additional, more specific attributes, would be simply the non-negative integer indicating the number of photons possessed of the aforementioned characteristics and present in the electromagnetic field whenever the compound quantum, electrodynamic system is in a quantum state associated with this set of values of all the occupation-number operators related to photons. In an analogous way, occupation-number operators are assigned to the aggregate of all the electrons and positrons present in the electromagnetic field. The initial and the final quantum states of the compound quantum electrodynamic system are uniquely determined by a set of particular values taken on by all the occupation-number operators associated with photons, electrons and positrons. However, the above condition implying specific values of all the occupation-number operators characterizes certain specific quantum states of the compound system but is not comprehensive enough to yield a general definition of a quantum state which we are trying to obtain. For in cases when the quantum state of the compound system is not an 'eigen-state' of some occupation-number operator, this operator will not have any particular value in this quantum state. The general definition of such a quantum electrodynamic state can be obtained by using the superposition of all quantum states which involve specific values of each occupation-number operator with the understanding that, in this superposition, every component quantum state is multiplied by an appropriate, complex coefficient.

In Feynman's formulation of quantum electrodynamics, the initial and final quantum states of the compound system constitute the boundary-conditions of the quantum electrodynamic process which provides a transition from the initial to the final quantum state. Each transitional process consists of a specific, temporal development of the constitutive components of radiation and matter in addition to the temporal development of the interaction-energy linking radiation and matter. Since the radiation field can be described by the vector and scalar electromagnetic potentials, a transitional process of the radiation-field will consist of any pre-assigned, functional dependency of each of these two potentials upon time, defined within the time-interval separating the initial and the final quantum states of the compound system. Similarly, a transitional process in the charged matter interacting with the radiation will consist of any choice of the functional dependencies upon time of all the spatial coordinates of all the electrons and positrons, during the same time-interval. Finally, a specific, transitional process of the interaction-energy is constituted by its functional dependency upon time generated by the functional dependency of this interaction-energy upon both electromagnetic potentials and the coordinates of all electrons and protons.

We have explained that the action involved in a transitional, quantum electrodynamic process is made up of three terms which correspond to the action of the electromagnetic radiation, the charged matter interacting with this radiation and the additional term which is determined by the time-integral of the interaction energy of radiation and matter. The fundamental axiom of quantum electrodynamics in Feynman's axiomatization of this theory determines the functional dependency of the probability-amplitude (and, consequently, also of the probability) of any transitional process which leads from the initial to the final state of the compound, quantum electrodynamic system, upon the total action involved in this process.

Needless to say, the primary objective of Feynman's 'spatio-temporal' axiomatization of quantum electrodynamics is physical vigor rather than mathematical rigor. It would make little sense, therefore, to interpret literally some of the claims he makes. Thus, according to his statement, the sole quantum electrodynamical axiom proposed by him is sufficient to determine the probability-amplitude for the transition from one quantum electrodynamic state to another one. This clashes with a statement made by Feynman to the effect that the term 'quantum electrodynamics', as used by him in the axiomatic system he proposes, deals only with the quantization of the pure, electromagnetic field and disregards the electrons and positrons in the field entirely. The interaction of the electrically charged matter with the field of electromagnetic radiation in which it is located is obviously one of the main achievements of Feynman's 'spatio-temporal' approach to quantum electrodynamics (Feynman, 1949b).

Another aspect of Feynman's theory is the role the 'Least Action Principle' plays in it. We have seen that this variational principle is not relevant to Feynman's quantum electrodynamical axiom, although the concept of physical action is relevant to this axiom. The minimum requirement on the total action of the compound quantum electrodynamic system serves other purposes, e.g., it proves sufficient for the derivation of Maxwell's equations of the electromagnetic field. Literally speaking, Feynman's theory contains, therefore, two distinct axioms which both involve the concept of physical action. One axiom condenses the law governing the transition-probabilities between consecutive states of the compound, quantum electrodynamic system. The other axiom states a minimum condition on the relevant physical actions and serves purposes that have little in common with the quantum electrodynamic axiom.

The logical difficulties inherent in contemporary quantum electrodynamics, including its presentation by Feynman, come from the infinitist divergencies of some crucial, quantum electrodynamic quantities

(Bogoliubov & Shirkov, 1959). These difficulties, which Feynman stresses and does not claim to solve, are not discussed here.

Finally, let me comment briefly on the contribution to the problem of indeterministic causality which Feynman has made in his various papers on the 'spatio-temporal approach to quantum theory'. I have only emphasized the remarkable contribution made by Feynman's spatio-temporal approach to the issue of indeterministic causality in quantum electrodynamics. In this case, his achievement comes from his two-fold use of the concept of physical action, namely a variational and a non-variational application of this idea.

The interesting point, however, is that Feynman has also applied the spatio-temporal approach to another indeterministic physical theory, namely to non-relativistic quantum mechanics (Feynman, 1956). In this case, he does not postulate at all that the physical action involved in quantum mechanical processes should be minimal. He succeeds, nevertheless, in deriving Schrödinger's partial differential equation by using a procedure which involves the concept of action not subjected to a minimum requirement. It seems to me that in the case of non-relativistic quantum mechanics, Feynman's novel, non-variational use of the concept of physical action sheds also a new light on indeterministic causality in the quantal theory under consideration, i.e. the quantum mechanics of Heisenberg and Schrödinger. The point is, that Maxwell's equations of the electromagnetic field derived by Feynman from a minimum requirement on physical action, are strictly deterministic. Within quantum mechanics, there is no deterministic counterpart for Maxwell's equations. This explains why the Least Action Principle is not literally applied in the spatio-temporal approach to non-relativistic quantum mechanics. It becomes also clear, from this point of view, that neither quantum mechanics nor quantum electrodynamics are deterministic: both theories are derived from a non-variational use of the idea of physical action.

To sum up the view of indeterministic causality which is inherent in Feynman's quantum electrodynamics. The physical action involved in a transitional quantum electrodynamic process consists of the three components corresponding to the energy of the electromagnetic field, of the electrically charged particles in the field and of the interaction-energy of field and matter. This action is not subject to any minimum requirement in the single quantum electrodynamic law, which specifies the functional dependency of the conditional probability of any state of the compound quantum electrodynamical system upon an earlier state of this system. The formulation of the law shows immediately that this conditional probability is always less than 1. Accordingly, this is the indeterministic extension of deterministic causality which prevails in quantum electrodynamics.

The scope of indeterministic causality in quantum electrodynamics was never investigated in any detail. So far, I have stated the principle of indeterministic causality which is valid in this area, only with regard to those special, comprehensive events which constitute quantum states of the quantum electrodynamic system. In the companion paper, I shall spell out the reasons why this principle of indeterministic causality applies also to less comprehensive, and even to point-like events of this area.

The relativistic nature of quantum electrodynamics, i.e., the invariance of the fundamental, quantum electrodynamical equations under the Lorentz-group of spatio-temporal transformations, have been explicitly proven (Heitler, 1960). In Feynman's formulation of this quantum theory no such tedious proof is necessary. The relativistic invariance is apparent from the wording of his 'only' quantum electrodynamical axiom.

I shall conclude the above concise analysis of a relativistic quantum theory, namely quantum electrodynamics, with a tentative formulation of the principle of indeterministic causality which apparently prevails in theories of this type. Only indeterministic extensions of individual, deterministic causality were considered. The problem of statistical, indeterministic causality in relativistic quantum theories will be discussed in my companion paper. The main point is, obviously, that the principle of indeterministic causality involving the concept of conditional probability is a straightforward consequence of Feynman's quantum electrodynamical axiom and that this concept of indeterministic causality applies to quantum electrodynamical systems regardless of their causal interaction with other systems. Consequently, the scope of indeterministic causality, restricted in non-relativistic quantum mechanics to non-interacting systems, is obviously broader in quantum electrodynamics, where no such restriction is valid. This holds also of the quantum theory of dispersion (Goldberger, 1960).

Moreover, the reason why quantum electrodynamics is governed by a principle of indeterministic causality, in contrast to the theories of classical physics, is illuminated by the different use made of the Least Action Principle in classical and in quantal physics. In the latter, the requirement of minimizing the physical action involved in a transition from an initial to the final state of the quantum electrodynamical system is irrelevant, although the probability of any transition of this sort depends upon the physical action required by any specific transition. In the former, the requirement of minimal action is used in order to uniquely determine the only possible transition under given boundary-conditions. This is, perhaps, the first rational explanation of the breakdown of deterministic causality in the aggregate of quantum theories.

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